$$
\begin{array}{ccc}
 & & \eta \text{ generic pt} \\
 & & \cap \\
f : X & \to & C \quad \text{proper flat} \\
\cup & & \in \\
X_c & & c \quad \text{closed pt} \\
\| & & \\
f^{-1}(c) & &
\end{array}
$$

w. C a sm curve

(1) (Esnault–Viehweg '85)

X_c is 2-dim klt

$\Rightarrow$ general fibers are klt

(2) (Ishii '86, '89)

- X_c is 2-dim lc

  $\Rightarrow$ so are general fibers

- If X_η is $\mathbb{Q}$-Gor,

  X_c has only isolated lc sings

  $\Rightarrow$ so do general fibers

(3) (Shepherd-Barron - Kollár '88)

If X_η is $\mathbb{Q}$-Gor. and X is normal,

X_c is 2-dim slc

$\Rightarrow$ so are general fibers

(4) (Kawamata '99, Nakayama '04)

X_c is term (resp. can.)

$\Rightarrow$ so are general fibers $\quad \boxed{2}$

<u>Q</u>

What if $\dim X_c \geq 3$

in (1) ~ (3) ?

<u>Rem</u>

- If X is $\mathbb{Q}$-Goren,

(1) ~ (3) holds in higher dim.

pf

X_c is a Cartier div on X since X is $\mathbb{Q}$-Gor.

by inv. of adj,

X_c is klt (lc)

$\Rightarrow$ X is klt (lc)

$\Rightarrow$ X_η is klt (lc)

$\Leftarrow$ general fibers are klt (lc) //

- If X is NOT $\mathbb{Q}$-Gor,

$\exists$ counter-example

Example (cf. Singh '99)

$$S = \mathbb{C}[a, b, c, d, e]$$

$$M = \begin{pmatrix} a^2+e^5 & b & d \\ c & a^2 & b^2-d \end{pmatrix}$$

$I := I_2(M)$ ideal gen by $\forall$ 2×2-minors of M $\boxed{4}$

$$
\begin{tikzcd}
I := I_2(M) \arrow[d, phantom, "\cap"] \\
S
\end{tikzcd}
$$

$X := V(I) \subset \operatorname{Spec} S = \mathbb{A}^5$

$\cup$

$D = V(e)$ 2-dim flat

$$\varphi : S \to T := \mathbb{C}[Y_1, \cdots, Y_6]$$

$$
\begin{aligned}
a &\longmapsto Y_1 \\
&\vdots \\
d &\longmapsto Y_4 \\
e &\longmapsto Y_5 Y_6
\end{aligned}
$$

$$
\begin{tikzcd}
\mathcal{X} := V(\varphi(I)) \subset \operatorname{Spec} T = \mathbb{A}^6 \arrow[d] \arrow[dr, "f"'] & \\
& \operatorname{Spec} \mathbb{C}[Y_6] = \mathbb{A}^1
\end{tikzcd}
$$

$\in$ 0

$\mathcal{X}_0 \cong D \times \mathbb{A}^1 \leftarrow$ 3-dim flat

$\mathcal{X}_t \cong X$ for $\forall t \neq 0$

X is NOT $\mathbb{Q}$-Gor. LS

$\Rightarrow$ X_t is NOT $\mathbb{Q}$-Gor.

in part, NOT Klt

<u>Q</u> What if X_η is $\mathbb{Q}$-Gor ?

X normal var. $/k$

$\pi : Y \to X$ proper birat w. Y normal

D Weil div on X

not nec $\mathbb{Q}$-Cartier

<u>Def</u>

$\pi^{\natural} D$ is the Weil div on Y

s.t. $\mathcal{O}_Y(-\pi^{\natural} D) = (\mathcal{O}_X(-D)\,\mathcal{O}_Y)^{**}$

reflexive hull

$$\pi^* D := \lim_{m \to \infty} \frac{\pi^{\natural}(mD)}{m}$$

$$= \inf_m \frac{\pi^{\natural}(mD)}{m}$$

$\mathbb{R}$-Weil div.

Def (de Fernex–Hacon '09, Chiecchio–Urbinati '18)

Assume π is a log resol.

$$K_{Y/X}^{-} := K_Y - \pi^* K_X$$

$$K_{Y/X}^{+} := K_Y + \pi^*(-K_X)$$

Note

- $K_{Y/X}^{+} \geq K_{Y/X}^{-}$

- If X is $\mathbb{Q}$-Gorenstein,

$$K_{Y/X}^{+} = K_{Y/X}^{-} = K_{Y/X}$$

X is klt type (lc) $\iff K_{Y/X}^{-} > -1$ $(\geq)$

X is valuatively klt (——— lc) $\iff K_{Y/X}^{+} > -1$ $(\geq)$

Note

- klt type (lc) $\Rightarrow$ vklt (vlc)

- If X is $\mathbb{Q}$-Gor,

  klt type (lc) = v klt (v lc) = klt (lc)

- In dim 2,

  klt type (lc) = v klt (lc) = klt (lc)

- X is klt type (lc) $\Longleftrightarrow$ $\exists\, \Delta \geq 0$ $\mathbb{Q}$-div s.t. $K_X + \Delta$ $\mathbb{Q}$-Cartier, (X, Δ) klt (lc)

- klt type sings are NOT inv under small deform.

In the previous example, X_0 is 3-dim klt ($\Longrightarrow$ klt type) but X_t is NOT klt type

<u>Thm</u> (Sato - T)

X normal, $D \subset X$ normal Cartier div

D is vKlt $\Rightarrow$ X is vKlt around D

Cor

$f : \mathcal{X} \to C$ proper flat, $c \in C$ w. C sm curve

If $\mathcal{X}_c$ is vKlt $\Rightarrow$ so are general fibers

In part, if $\mathcal{X}_\eta$ is $\mathbb{Q}$-Goren.

$\mathcal{X}_c$ is Klt $\Rightarrow$ so are general fibers

Pf of Thm

Use multiplier modules $\mathcal{J}(\omega_X, \mathfrak{a}^t)$

$\mathfrak{a}$ frac ideal, $t > 0$

$\pi : Y \to X$ log resol of X

fixed s.t. $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$ inv.

$\mathcal{J}(\omega_X, \mathfrak{a}^t) = \pi_* \mathcal{O}_Y(K_Y - \lfloor tF \rfloor))$

Γ Cartier div on X containing Sing X

X is vKlt

$\Leftrightarrow \mathcal{J}(\omega_X, \mathcal{O}_X(-\Gamma)^{\varepsilon}\mathcal{O}_X(mK_X)^{1-\frac{1}{m}}) = \mathcal{O}_X(mK_X)$ for $\forall m$ $0<^{\forall}\varepsilon \ll 1$

$\subset$ always holds

May assume
- X is affine
- $-K_X \geq 0$
- K_X and D have no common comp.

Fix Γ containing Sing X and Sing D.

D vKlt

$$\Rightarrow \underbrace{\mathcal{J}(\omega_D, \mathcal{O}_D(-\Gamma|_D)^{\varepsilon}(\mathcal{O}_X(mK_X)\mathcal{O}_D)^{1-\frac{1}{m}})}_{I_D :=} \supset \mathcal{O}_X(mK_X)\mathcal{O}_D$$

$(\mathcal{O}_X(mK_X)\mathcal{O}_D \subset \mathcal{O}_D(mK_D))$

We use the restriction thm for multiplier modules

$$I_D < \underbrace{\mathcal{J}(\omega_X, \mathcal{O}_X(-n)^{\varepsilon}\mathcal{O}_X(mK_X)^{1-\frac{1}{m}})}_{\substack{\| \\ I_X}} \qquad \mathcal{O}_D$$

$\boxed{/0}$

Using Nakayama's lem,

$$I_X \supset \mathcal{O}_X(mK_X)$$

$\Rightarrow$ X is ~~vlt~~ vlc.

<u>Thm</u> (Sato - T)

X normal, $D \subset X$ normal Cartier (S2 and G1) div

If D is $\mathbb{Q}$-Gor. lc $\Rightarrow$ X is vlc (slc)

<u>Conj</u>

If D is vlc (vslc) $\Rightarrow$ X is vlc.

Cor $f : X \to C$ flat proper

$c \in C$

(1) If X_c is $\mathbb{Q}$-Gor lc

$\Rightarrow$ general fibers are vlc.

(2) If $X_?$ is $\mathbb{Q}$-Gorn.

X_c is lc $\Rightarrow$ so are general fibers

pf

Step 1 characterize vlc sings

in terms of Fujino's non-lc ideals.

a variant of

$$\mathcal{J}_{NLC}(\omega_X, \mathfrak{a}^t) = \pi_* \mathcal{O}_Y(K_Y - \lfloor t F \rfloor + \underbrace{\sum E_i}_{\text{reduced } \pi\text{-exc div.}})$$

Step 2

Use restriction thm

(this follows from vanishing thm

for lc sings due Ambro and Fujino ...)

Rem

Kollár proved Cor (2) using lc modification of X.

X proj normal var.

$$Z = \mathrm{Spec} \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nH))$$

(H ample)

(lc)

Z is klt type

$\Updownarrow$

X is Fano type

CY

klt type $\Rightarrow$ CM $\not\Leftarrow$ vklt